Swan conductors on the boundary of Lubin-Tate spaces

Stefan Wewers

Abstract

Lubin-Tate spaces of dimension one are finite étale covers of the non-archimedian open unit disk. We compute certain invariants which measure the ramification of this cover over the boundary of the disk.

1 Introduction

1.1 Let F be a local field, i.e. a finite extension of $\mathbb{Q}_p$ or the field of Laurent series over a finite field. Let $\mathcal{O}$ denote the ring of integers of F. Choose a uniformizer ϖ of $\mathcal{O}$. The residue field $k = \mathcal{O}/(\varpi)$ is a finite field with, say, q elements. We let $\hat{F}^{\rm nr}$ denote the completion of the maximal unramified extension of F and $\hat{\mathcal{O}}^{\rm nr} \subset \hat{F}^{\rm nr}$ its ring of integers. Fix an integer $h \geq 1$.

A construction due to Lubin-Tate and Drinfeld attaches to F and h a certain inverse system $\cdots \to \mathcal{X}(\varpi^2) \to \mathcal{X}(\varpi) \to \mathcal{X}(1)$ of formal schemes $\mathcal{X}(\varpi^n) = {\rm Spf} A_n$, called the *Lubin-Tate tower*. In this paper we only consider the case $h = 2$, and we look at the tower of rigid analytic spaces

$$\mathsf{X}(\varpi^n) := \mathcal{X}(\varpi^n) \otimes_{\hat{\mathcal{O}}^{\rm nr}} \hat{F}^{\rm nr}, \quad n = 1, 2, \ldots$$

associated to $(\mathcal{X}(\varpi^n))_n$. At the lowest level, $\mathsf{X}(1)$ is isomorphic to the open unit disk over the non-archimedian field $\hat{F}^{\rm nr}$. The maps $\mathsf{X}(\varpi^n) \to \mathsf{X}(1)$ are finite étale Galois covers with Galois group ${\rm GL}_2(\mathcal{O}/(\varpi^n))$. Therefore, the rigid space $\mathsf{X}(\varpi^n)$ is a smooth analytic curve.

The present paper is concerned with computing certain invariants which measure the ramification of the étale cover $\mathsf{X}(\varpi^n) \to \mathsf{X}(1)$ over the 'boundary' of the open disk $\mathsf{X}(1)$. These computations are a key ingredient for the results of [20], which describe the *stable reduction* of $\mathsf{X}(\varpi^n)$.

Our notation $\mathsf{X}(\varpi^n)$ (which is not standard) suggests an analogy with modular curves. And indeed, for $F = \mathbb{Q}_p$ the connected components of $\mathsf{X}(p^n) \otimes_{\mathbb{Q}_p^{\rm nr}} \mathbb{C}_p$ are isomorphic to certain open analytic subspaces of the classical modular curve $X(p^n m)$ associated to the full congruence subgroup of ${\rm SL}_2(\mathbb{Z})$ modulo $p^n m$, for $m \geq 3$ and prime to p. Each of these analytic subspaces is the formal fiber of a singularity of the p-adic integral model of $X(p^n m)$ studied by Katz and Mazur [14]. Combining the results of [20] with the results of Katz and Mazur yields,

in the special case $F = \mathbb{Q}_p$, a description of the stable reduction of $X(p^n m)$ at the prime p. There is also an interesting connection with the local Langlands correspondence. A theorem of Carayol (generalized by Harris and Taylor to the case $h > 2$) asserts that the étale cohomology of the tower $\varprojlim \mathsf{X}(\varpi^n)$ realizes the local Langlands correspondence for the group $\mathrm{GL}_2(F)$. Using the description of the stable reduction of $\mathsf{X}(\varpi^n)$ in [20], it is possible to give a new and more direct proof of this theorem (work in progress).

1.2 To explain the result of this paper and their relevance for the results of [20], we abstract a bit from the special situation of Lubin-Tate spaces. Let K be a complete discrete non-archimedian field. Let R denote the ring of integers of K. Let Y denote the open unit disk over K and let $f : \mathsf{X} \to \mathsf{Y}$ be a finite étale Galois cover with Galois group G. In what follows, we shall always allow the field K to be replaced by a suitable finite extension. (We do not replace K by its algebraic closure because we want the valuation to be discrete. See [21].)

By the semistable reduction theorem, the rigid space X has a minimal semistable formal model $\mathcal{X}$ over R. Let $\mathcal{Y} := \mathcal{X}/G$ be the quotient by the G-action; this is a semistable formal model of Y. Such a semistable model of the disk Y is easy to describe: it is determined by a finite collection of closed affinoid disks $\mathsf{D}_i \subset \mathsf{Y}$. Namely, each closed disk D_i determines a blowup of the standard formal model $\mathrm{Spf} R[[T]]$ of Y, and performing all these blowups simultaneously yields the semistable model $\mathcal{Y}$. We say that the disks D_i are *relevant* for the stable reduction of X. In some sense, the main problem in finding the stable reduction of X is to find the relevant disks D_i. See [21] for more details.

Let τ be an irreducible representation of the group G. We assume that τ is defined over a finite extension of $\mathbb{Q}_\ell$, where ℓ is an auxiliary prime dividing neither the order of G nor the characteristic of the field k. The representation τ and the Galois cover $f : \mathsf{X} \to \mathsf{Y}$ determine a lisse $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on the étale topology of Y.

Let $x \in \mathsf{Y}$ be a closed point. For a positive rational number $s \in \mathbb{Q}_{>0}$ we let $\mathsf{D}(x, s) \subset \mathsf{Y}$ denote the closed affinoid disk with center x and radius $r := |\varpi|^s$. Following Huber [13] and Ramero [18], we define two numbers,

$$\mathrm{sw}_{\mathcal{F}}(s) \in \mathbb{Z}_{\geq 0}, \qquad \delta_{\mathcal{F}}(s) \in \mathbb{Q}_{\geq 0},$$

which we call the *Swan conductor* and the *discriminant conductor*. These numbers measure, in some sense, the ramification of the sheaf $\mathcal{F}$ over the disk $\mathsf{D}(x, s)$. For instance, the discriminant conductor $\delta_{\mathcal{F}}(s)$ is essentially the classical Swan conductor of the Galois representation associated to $\mathcal{F}$ and the discrete valuation of the field $\mathrm{Frac}(R[[T]])$ corresponding to the maximum norm on $\mathsf{D}(x, s)$.

The functions $s \mapsto \mathrm{sw}_{\mathcal{F}}(s)$ and $s \mapsto \delta_{\mathcal{F}}(s)$ extend to functions on the interval $[0, \infty) \subset \mathbb{R}$ with the following properties: (a) $\mathrm{sw}_{\mathcal{F}}$ is right continuous, locally constant and decreasing, (b) $\delta_{\mathcal{F}}$ is continuous and piecewise linear, and (c) $\mathrm{sw}_{\mathcal{F}}$ is equal to minus the right derivative of $\delta_{\mathcal{F}}$,

$$\frac{\partial}{\partial s}\delta_{\mathcal{F}}(s+) = -\mathrm{sw}_{\mathcal{F}}(s).$$

It follows that $\delta_{\mathcal{F}}$ is convex and decreasing and that both $\mathrm{sw}_{\mathcal{F}}$ and $\delta_{\mathcal{F}}$ are $\equiv 0$ for $s \gg 0$. Furthermore, there are are finite number of (rational) critical values $s_1, \ldots, s_n$ where $\mathrm{sw}_{\mathcal{F}}$ is discontinuous and where $\delta_{\mathcal{F}}$ is not smooth. We call $s_1, \ldots, s_n$ the *breaks* of $\delta_{\mathcal{F}}$.

The study of the function $\delta_{\mathcal{F}}$ goes back to Lütkebohmert's paper [17] on the p-adic Riemann's Existence Theorem. (In *loc.cit* one studies finite étale covers $f : \mathsf{X} \to \mathsf{Y}^* := \mathsf{Y} - \{0\}$ of the *punctured* disk, and the representation τ is the regular representation of G. In this case the integers $\mathrm{sw}_{\mathcal{F}}(s)$ may become negative, and $\delta_F(s)$ may never be zero.) To prove the properties of the function $\delta_{\mathcal{F}}$ mentioned above one uses the semistable reduction theorem. As an immediate consequence of this proof, one obtains the following characterization of the breaks $s_1, \ldots, s_n$: the disk $\mathsf{D}(x, s_i)$ are relevant for the stable reduction of the cover $f : \mathsf{X} \to \mathsf{Y}$. This is the reason why we are interested in studying the function $\delta_{\mathcal{F}}$.

1.3 To apply the previous discussion to the Lubin-Tate tower, we set $\mathsf{Y} := \mathsf{X}(1)$, $\mathsf{X} := \mathsf{X}(\varpi^n)$ and $G = G_n := \mathrm{GL}_2(\mathcal{O}/\varpi^n)$. We also have to choose a point $x \in \mathsf{X}(1)$ (the center for the disks we look at) and a representation τ of G_n (which gives rise to the sheaf $\mathcal{F}$). It turns out that, in order to describe the semistable reduction of $\mathsf{X}(\varpi^n)$, it suffices to choose the following pairs (x, τ):

- The point $x \in \mathsf{X}(1)$ is a *canonical point* corresponding to a separable quadratic extension E/F (in the case $F = \mathbb{Q}_p$ such a point gives rise to a CM-point (of a certain kind) on the corresponding modular curve).

- The restriction of the representation τ to the compact group $K := \mathrm{GL}_2(\mathcal{O})$ is a *type* for an irreducible supercuspidal representation of $\mathrm{GL}_2(F)$. Moreover, the quadratic extension associated to τ by the classification of such types (see e.g. [15] or [3]) is the extension E/F.

For each pair (x, τ) as above, one can explicitly compute the function $\delta_{\mathcal{F}}$ and, in particular, determine its breaks $s_1, \ldots, s_r$. As we have remarked above, the disks $\mathsf{D}(x, s_i)$ are all relevant for the stable reduction of $\mathsf{X}(\varpi)$. It is not true that every relevant disk occurs in this way, but nevertheless, knowing these ones is sufficient to study the stable reduction of $\mathsf{X}(\varpi^n)$. See [20].

We hope that these explanations sufficiently motivate the main result of this paper. It computes the value of functions $\mathrm{sw}_{\mathcal{F}}$ and $\delta_{\mathcal{F}}$ at the point $s = 0$ in all the cases we are interested in.

Theorem 1.1 *Let τ be the K-type of an irreducible supercuspidal representation of $\mathrm{GL}_2(F)$. Assume that τ is minimal of level n, and consider τ as a representation of $G_n = \mathrm{GL}_2(\mathcal{O}/\varpi^n)$. (See §4.2 for the terminology.) Let $\mathcal{F}$ be the sheaf on $\mathsf{X}(1)$ corresponding to τ. Let E/F be the quadratic extension associated to τ by the construction in [15]. Let $x \in \mathsf{X}(1)$ be a canonical point, corresponding to the extension E/F (actually, any point $x \in \mathsf{X}(1)$ would do). If the extension E/F is unramified then we have*

$$\mathrm{sw}_{\mathcal{F}}(0) = -(q+1)q^{n-1}, \quad \delta_{\mathcal{F}}(0) = (nq - n + 1)q^{n-1}.$$

If E/F is ramified then

$$\mathrm{sw}_{\mathcal{F}}(0) = -(q+1)q^{n-2}, \quad \delta_{\mathcal{F}}(0) = (nq - q - n)q^{n-2}.$$

Note that, a priori, the values of $\mathrm{sw}_{\mathcal{F}}$ and $\delta_{\mathcal{F}}$ at $s = 0$ are not defined in the same way as for rational values $s > 0$, simply because the closed unit disk $D(x, 0)$ is not contained in the open unit disk $\mathsf{X}(1)$. There are essentially two ways to get hold of the value at $s = 0$. Firstly, one can try to extend the cover $f : \mathsf{X}(\varpi^n) \to \mathsf{X}(1)$ to the closed unit disk. This is possible because the cover f essentially occurs inside an algebraic cover between certain Shimura or Drinfeld modular curves (classical modular curves for $F = \mathbb{Q}_p$). However, this is not very helpful to actually compute $\mathrm{sw}_{\mathcal{F}}(s)$ and $\delta_{\mathcal{F}}(s)$ for $s = 0$. So we follow the second possibility, i.e. we extend the definition of $\mathrm{sw}_{\mathcal{F}}$ and $\delta_{\mathcal{F}}$ in a way that allows us to compute these values for $s = 0$ directly in terms of the rank-two valuation of $\mathrm{Frac}(R[[T]])$ corresponding to the 'boundary' of the open unit disk.

2 The Swan and the discriminant conductor

We introduce Swan and discriminant conductors for étale Galois covers of so-called *open analytic curves*, extending results of Huber [13] and Ramero [18], where this is done for smooth affinoid curves.

2.1 Recall: open analytic curves We fix a field K_0 which is complete with respect to a discrete non-archimedian valuation $|\cdot|$, and whose residue field k is algebraically closed and of positive characteristic $p > 0$. We choose an algebraic closure K^{ac} of K_0 and extend the valuation $|\cdot|$ to K_0^{ac}. We let Γ denote the value group of the valuation $|\cdot|$ on K^{ac}. We assume that Γ is a subgroup of $\mathbb{R}_{>0}$.

We recall the following definitions from [21].

Definition 2.1 An *open analytic curve* is given by a pair (K, X), where $K \subset K_0^{\mathrm{ac}}$ is a finite extension of K_0 and X is a rigid analytic space over K. We demand that X is isomorphic to $C - D$, where C is the analytification of a smooth projective curve over K and $D \subset C$ is an affinoid subdomain intersecting every connected component of C.

A morphism between two open analytic curves (K_1, X_1) and (K_2, X_2) is an element of the direct limit

$$\mathrm{Hom}(\mathsf{X}_1, \mathsf{X}_2) := \varinjlim_{K_3} \mathrm{Hom}(\mathsf{X}_1 \otimes K_3, \mathsf{X}_2 \otimes K_3),$$

where $K_3 \subset K_0^{\mathrm{ac}}$ ranges over all common finite extensions of K_1 and K_2.

We shall follow the same convention as in [21], regarding the field K. That is, we simply write X to denote an open analytic curve. The field K is only mentioned if needed, and it is always assumed to be 'sufficiently large'. For instance, if we say that X is connected then this means that $\mathsf{X} \otimes_K K'$ is connected, for every finite extension K'/K.

Definition 2.2 An *underlying affinoid* is an affinoid subdomain $\mathsf{U} \subset \mathsf{X}$ such that $\mathsf{X} - \mathsf{U}$ is the disjoint union of open annuli none of which is contained in an affinoid subdomain of X. An *end* of X is an element of the inverse limit of the set of connected components of $\mathsf{X} - \mathsf{U}$, where U ranges over all underlying affinoids. The set of all ends is denoted by $\partial\mathsf{X}$.

2.2 The rank two valuation associated to an end Let X be an open analytic curve with field of definition K. We let R denote the valuation ring of K. Let $x \in \partial\mathsf{X}$ be an end of X. We will associate to the pair (X, x) a certain field K_x equipped with a rank two valuation $|\cdot|_x$. The construction should depend only on the connected component of X on which x lies, so we may from the start assume that X is connected.

Recall from [21] that X has a certain canonical formal model $\mathcal{X}$ called the *minimal model*, as follows. Let $\mathcal{O}_{\mathsf{X}}^\circ$ denote the ring of power bounded analytic functions on X. According to [2], $\mathcal{O}_{\mathsf{X}}^\circ$ is a normal and complete local ring, isomorphic to the completion of an R-algebra of finite type. We set $\mathcal{X} := \operatorname{Spf} \mathcal{O}_{\mathsf{X}}^\circ$; then the generic fiber of $\mathcal{X}$ can be canonically identified with X. By choosing the field K sufficiently large we may assume that the scheme $\mathcal{X}_s := \operatorname{Spec}(\mathcal{O}_{\mathsf{X}}^\circ \otimes k)$ is reduced. Under this assumption, the construction of $\mathcal{X}$ is stable under further extension of K, i.e. if K'/K is a finite extension with valuation ring R' then $\mathcal{X}' = \mathcal{X} \otimes_R R'$ is the minimal model of $X \otimes_K K'$.

There is a natural bijection between the ends of X and the generic points of the scheme $\mathcal{X}_s$. We let $\eta \in \mathcal{X}_s$ denote the generic point corresponding to $x \in \partial\mathsf{X}$. To η corresponds a discrete valuation of the fraction field of $\mathcal{O}_{\mathsf{X}}^\circ$ which extends the valuation $|\cdot|$ on K. We extend this valuation to the field $\operatorname{Frac}(\mathcal{O}_{\mathsf{X}}^\circ \otimes_R K^{\rm ac})$ in the obvious way and denote it by $|\cdot|_\eta$. The residue field $k(\eta)$ of $|\cdot|_\eta$ is a field of Laurent series, $k(\eta) \cong k((t))$. We define the rank two valuation $|\cdot|_x$ on $\operatorname{Frac}(\mathcal{O}_{\mathsf{X}}^\circ \otimes_R K^{\rm ac})$ as the composition of $|\cdot|_\eta$ with the canonical valuation on the residue field $k(\eta)$ (see e.g. [22]). We define the field

$$K_x := \operatorname{Frac}(\mathcal{O}_{\mathsf{X}}^\circ \otimes_R K^{\rm ac})^\frown$$

as the completion of $\operatorname{Frac}(\mathcal{O}_{\mathsf{X}}^\circ \otimes_K K^{\rm ac})$ with respect to $|\cdot|_x$. One easily checks that the field K_x is henselian.

Notation 2.3 Let Γ_x denote the valuation group of $|\cdot|_x$. Then

$$\Gamma_x = \Gamma \times \Lambda_x, \quad \Lambda_x = \langle \gamma_x \rangle, \tag{1}$$

where Λ_x is an ordered cyclic group generated by an element $\gamma_x < 1$. The ordering on Γ_x is lexicographic, i.e. for $r, s \in \Gamma$ and $i, j \in \mathbb{Z}$ we have $(r, \gamma_x^i) < (s, \gamma_x^j)$ if either $r < s$, or if $r = s$ and $i > j$. Let $|\cdot|_x^\flat$ denote the rank one valuation on K_x which is the composition of $|\cdot|_x$ with the first projection $(r, \gamma_x^i) \mapsto r$ (this was denoted $|\cdot|_\eta$ before). Let $\# : \Gamma_x \to \mathbb{Z}$ be the group homomorphism $(r, \gamma_x^i) \mapsto i$ and write $\#_x$ for the composition of $|\cdot|_x$ with $\#$.

The splitting (1) of Γ_x is canonical. To see this, note that we have identified Γ with the maximal divisible subgroup of Γ_x. Furthermore, the generator γ_x of Λ_x is the maximal element of Γ_x which is strictly smaller than 1. An element $u \in K_x$ such that $|u|_x = \gamma_x$ is called a *parameter* for the end x.

Example 2.4 Let $\epsilon \in \Gamma$, $\epsilon < 1$. Choose a finite extension K/K_0 such that $\epsilon \in |K^\times|$. We regard the standard open annulus

$$\mathsf{X} := \mathsf{C}(\epsilon, 1) = \{\, u \mid \epsilon < |u| < 1 \,\}$$

as an open analytic curve with field of definition K. Clearly, X has two ends. We let $x \in \partial\mathsf{X}$ be the end corresponding to the family of open annuli $\mathsf{C}(\epsilon', 1) \subset \mathsf{X}$ for $\epsilon < \epsilon' < 1$. Choose an element $\pi \in R$ with $|\pi| = \epsilon$. Then

$$\mathcal{O}_{\mathsf{X}}^\circ = R[[\, u, v \mid uv = \pi \,]].$$

An element of $\mathcal{O}_{\mathsf{X}}^\circ$ can be written as a Laurent series $\sum_i c_i u^i$ with $c_i \in K$ such that $|c_i| \leq 1$ for $i \geq 0$ and $|c_i| \leq \epsilon^{-i}$ for $i < 0$. It follows that every element of $\mathrm{Frac}(\mathcal{O}_{\mathsf{X}}^\circ \otimes_R K^{\mathrm{ac}})$ can also be written as a Laurent series whose coefficients all lie in a finite extension of K and satisfy certain growth conditions. With this notation, we have

$$|\sum_i c_i u^i|_x = \max_{i \in \mathbb{Z}} (|c_i|, \gamma_x^i).$$

In other words, $|\sum_i c_i u^i|_x^\flat = \max_i |c_i|$, and $\#(\sum_i c_i u^i)$ is the first index i where $|c_i|$ takes its maximal value.

2.3 Functoriality Let $f : \mathsf{X} \to \mathsf{Y}$ be an analytic map between open analytic curves. Given $x \in \partial\mathsf{X}$ and $y \in \partial\mathsf{Y}$, the notation $f(x) = y$ means the following. For every sufficiently small open annulus $\mathsf{A} \subset \mathsf{X}$ representing the end x the restriction of f to A is finite onto its image, and $f(\mathsf{A}) \subset \mathsf{Y}$ is an open annulus representing y. We shall also write $f : (\mathsf{X}, x) \to (\mathsf{Y}, y)$ for a map f satisfying the above condition.

Proposition 2.5 *The map $f : (\mathsf{X}, x) \to (\mathsf{Y}, y)$ induces a finite extension of valued fields*

$$f_x^* : K_y \hookrightarrow K_x.$$

For all $u \in K_y^\times$ we have

$$|f_x^* u|_x^\flat = |u|_y^\flat$$

and

$$\#_x(f_x^* u) = [K_y : K_x] \cdot \#_y(u).$$

The index $[K_y : K_x]$ is equal to the degree of $f|_{\mathsf{A}} : \mathsf{A} \to f(\mathsf{A})$, where $\mathsf{A} \subset \mathsf{X}$ is a sufficiently small annulus representing the end x.

Proof: We prove the proposition first in two special cases.

Case 1: X is an open annulus, $f : \mathsf{X} \hookrightarrow \mathsf{Y}$ is an open immersion and $\mathsf{X} \subset \mathsf{Y}$ represents the end y.

By [21], §1, we can represent Y as the formal fiber $]z[_Y$, where $Y = \operatorname{Spec} A$ is an affine normal curve over R with reduced special fiber Y_s and $z \in Y_s$ is a closed point. The end y corresponds to a branch η of Y_s through z. Here we consider the branch η as a generic point of $\operatorname{Spec}(\hat{\mathcal{O}}_{Y,z} \otimes k)$. Let $Z \subset Y_s$ denote the irreducible component whose generic point is the image of η on Y_s.

Furthermore, we can identify the open subspace $\mathsf{X} \subset \mathsf{Y}$ with the formal fiber $]w[_X$, where $g : X \to Y$ is an admissible blowup with center z and $w \in X_s$ is a closed point. The assumption that X is an open annulus representing the end y implies that w is a formal double point, point of intersection of the strict transform of Z with the exceptional divisor of g. Let $\nu \in \operatorname{Spec}(\hat{\mathcal{O}}_{X,w} \otimes k)$ denote the branch of X_s through w corresponding to the strict transform of Z. To prove the proposition in Case 1 it suffices to show that the map

$$g_x^* : \mathcal{O}_{\mathsf{Y}}^\circ = \hat{\mathcal{O}}_{Y,z} \longrightarrow \mathcal{O}_{\mathsf{X}}^\circ = \hat{\mathcal{O}}_{X,w}$$

induced by g is compatible with the valuations $|\cdot|_y$ and $|\cdot|_x$ (i.e. $|g_x^* u|_x = |u|_y$) and induces an isomorphism $K_y \cong K_x$.

The valuations $|\cdot|_y^\flat$ and $|\cdot|_x^\flat$, restricted to $\mathrm{Frac}(\hat{\mathcal{O}}_{Y,z})$ and $\mathrm{Frac}(\hat{\mathcal{O}}_{X,w})$, are discrete and correspond to the codimension one points $\eta \in \operatorname{Spec}(\hat{\mathcal{O}}_{Y,z})$ and $\nu \in \operatorname{Spec}(\hat{\mathcal{O}}_{X,w})$. Since $g(\nu) = \eta$ and X_s and Y_s are reduced we have $|g_x^* u|_x^\flat = |u|_y^\flat$ for all $u \in \hat{\mathcal{O}}_{Y,z}$. The functions $\#_y$ and $\#_x$, restricted to $\mathrm{Frac}(\hat{\mathcal{O}}_{Y,z})^\times$ and $\mathrm{Frac}(\hat{\mathcal{O}}_{X,w})^\times$, are induced from the natural discrete valuations on the residue fields of η and ν. It is also clear that g_x^* induces an isomorphism between these two residue fields. Therefore, $|g_x^* u|_x = |u|_y$ holds for all $u \in \hat{\mathcal{O}}_{Y,z}$. It remains to show that g_x^* induces an isomorphism $K_y \cong K_x$. Actually, it suffices to show that $\hat{\mathcal{O}}_{X,w} \subset g_x^* K_y$. Let $u \in \mathcal{O}_{Y,z}$ be a parameter for y, i.e. an element with $|u|_y = \gamma_y$. By the definition of the valuation $|\cdot|_y$, u lies in the maximal ideal of $\hat{\mathcal{O}}_{Y,z}$, so we have $R[[u]] \subset \hat{\mathcal{O}}_{Y,z}$. Identify the ring $R[[u]]$ with its image in $\hat{\mathcal{O}}_{X,w}$ via g_x^*. Then u is also a parameter for $|\cdot|_x$. Therefore,

$$\hat{\mathcal{O}}_{X,w} = R[[\, u, v \mid uv = \pi \,]],$$

where $\pi \in R$ is a suitable element with $\epsilon := |\pi| < 0$. Consider $v = \pi/u$ as an element in K_y. Using

$$|v^i|_y = (\epsilon^i, \gamma_y^{-i}) \longrightarrow 0$$

for $i \to \infty$, we conclude that $\hat{\mathcal{O}}_{X,w}$ is contained in K_y, as desired. This completes the proof of the proposition in Case 1.

Case 2: X and Y are open annuli and f is a finite morphism.

In this case the induced map $f^* : \mathcal{O}_{\mathsf{Y}}^\circ \to \mathcal{O}_{\mathsf{X}}^\circ$ is a finite extension of normal local rings. With similar arguments as in Case 1, one shows that it induces an extension of valued fields

$$f_x^* : (\mathrm{Frac}(\mathcal{O}_{\mathsf{Y}}^\circ \otimes_R K^{\mathrm{ac}}), |\cdot|_y) \to (\mathrm{Frac}(\mathcal{O}_{\mathsf{X}}^\circ \otimes_R K^{\mathrm{ac}}), |\cdot|_x)$$

such that $|g_x^* u|_x^\flat = |u|_y^\flat$. It is also easy to see that the induced extension of residue fields of $|\cdot|_y$ and $|\cdot|_x$ (which are discretely valued fields) is totally ramified of degree $\deg(f)$. It follows that the induced extension $K_y \to K_x$ on the completions is of the same degree and has all the claimed properties. This proves the proposition in Case 2.

The general case of the proposition follows easily from Case 1 and Case 2. $\square$

2.4 Higher ramification groups Let X be an open analytic curve and $x \in \partial\mathsf{X}$ an end. Let G be a finite group acting faithfully on X and fixing x. By Proposition 2.5 the action of G on X induces an action of G on the valued field $(K_x, |\cdot|_x)$. It is easy to see that G acts faithfully on K_x. Following Huber [13] we define a filtration $(G_h)_{h\in\Gamma_x}$ of higher ramification groups on G.

For any $h \in \Gamma_x$, we set

$$G_h := \{\, \sigma \in G \mid h_x(\sigma) \leq h \,\},$$

where

$$h_x(\sigma) := \min\{\, h \mid |u - \sigma(u)|_x \leq h \cdot |u|_x \,\forall u \in K_x \,\}.$$

It is shown in [13], Lemma 2.1, that

$$h_x(\sigma) = \Big|\frac{t - \sigma(t)}{t}\Big|_x, \tag{2}$$

where $t \in K_x$ is any *parameter* at x, i.e. an element of K_x with $|t|_x = \gamma_x$. It is easy to see that the group

$$P := \cup_{h<1} G_h$$

is the maximal p-subgroup and that G/P is cyclic of order prime to p. Let

$$h_1 > \cdots > h_l$$

be the elements of the set $\{h(\sigma) \mid \sigma \in G, \sigma \neq 1\}$ which are $\neq 1$. Set $h_0 := 1$. By definition we have

$$G = G_{h_0} \supsetneq G_{h_1} \supsetneq \ldots \supsetneq G_{h_l} \supsetneq \{1\}.$$

The elements $h_i \in \Gamma_x$ for $i \geq 1$ are called the *jumps* in the filtration $(G_h)_h$.

There is also an upper numbering. Let $\mathsf{Y} := \mathsf{X}/G$ and let $y \in \partial\mathsf{Y}$ be the image of x. By Proposition 2.5 the extension K_x/K_y is of degree $|G|$. Since G acts faithfully on K_x and fixes K_y, the extension K_x/K_y is actually Galois, with Galois group G. Let $\varphi_{K_x/K_y} : \Gamma_x \otimes \mathbb{Q} \to \Gamma_y \otimes \mathbb{Q}$ be the function defined in [13], §2. For $\gamma \in \Gamma_y \otimes \mathbb{Q}$ set $G^\gamma := G_h$, where $h := \varphi^{-1}_{K_x/K_y}(\gamma)$. The elements

$$\gamma_i := \varphi_{K_x/K_y}(h_i) \in \Gamma_y \otimes \mathbb{Q}, \quad i = 1, \ldots, r,$$

are called the *jumps in the upper numbering*. Explicitly, we have

$$\gamma_i^\flat = \prod_{j=1}^{l} \Big(\frac{h_j^\flat}{h_{j-1}^\flat}\Big)^{|G_{h_j}|}, \quad \#\gamma_i = \sum_{j=1}^{i} \frac{\#h_j - \#h_{j-1}}{(G : G_{h_j})}, \tag{3}$$

with $h_j = (h_j^\flat, \#h_j)$ and $\gamma_j = (\gamma_j^\flat, \#\gamma_j)$.

2.5 The Swan conductor and the discriminant conductor We fix an open analytic curve Y and a prime number ℓ different from $p = \mathrm{char}(k)$. Let $\mathcal{F}$ be a lisse sheaf of $\bar{\mathbb{Q}}_\ell$-vectorspaces on Y. We say that $\mathcal{F}$ is *admissible* if there exists a finite group G of order prime to ℓ, an étale G-torsor $f : \mathsf{X} \to \mathsf{Y}$ and a representation τ of G on a finite-dimensional $\bar{\mathbb{Q}}_\ell$-vector space W such that

$$\mathcal{F} \cong (f_*\bar{\mathbb{Q}}_\ell)[\tau] := \underline{\mathrm{Hom}}_G(\underline{W}, f_*\bar{\mathbb{Q}}_\ell).$$

See [21] for more details and arguments why it makes sense to work with $\bar{\mathbb{Q}}_\ell$-coefficients in this situation.

Let $\mathcal{F}$ be an admissible sheaf on Y and $y \in \partial Y$ an end. We will attach to $\mathcal{F}$ and y two invariants, the *Swan conductor* $\mathrm{sw}_y(\mathcal{F}) \in \mathbb{Z}$ and the *discriminant conductor* $\delta_y(\mathcal{F}) \in \mathbb{R}$.

Choose an étale G-torsor $f : \mathsf{X} \to \mathsf{Y}$ and a representation $\tau : G \to \mathrm{GL}(V)$ such that $\mathcal{F} \cong (f_*\bar{\mathbb{Q}}_\ell)[\tau]$. Choose also an end $x \in \partial\mathsf{X}$ with $f(x) = y$. Let $G_x \subset G$ denote the stabilizer of x. In the previous subsection we defined a function $\sigma \mapsto h_x(\sigma)$ on G_x with values in Γ_x. We now define

$$\mathrm{sw}_x(\sigma) := -\#h_x(\sigma) \quad \text{if } \sigma \neq 1, \quad \mathrm{sw}_x(1) := \sum_{\sigma\neq 1} \#h_x(\sigma)$$

and

$$\mathrm{sw}_y := \mathrm{Ind}_{G_x}^G \mathrm{sw}_x.$$

This is a class function on G with values in $\mathbb{Z}$ and does not depend on the choice of x. By [13], Theorem 4.1, sw_y is a virtual character of G. Therefore,

$$\mathrm{sw}_y(\mathcal{F}) := \langle \mathrm{sw}_y, \tau\rangle_G$$

is an integer, which we call the *Swan conductor* of $\mathcal{F}$ at x. By [13], Proposition 4.2 (ii), this definition depends only on $\mathcal{F}$ but not on the chosen representation $\mathcal{F} \cong (f_*\bar{\mathbb{Q}}_\ell)[\tau]$.

It will be useful to have a formula for $\mathrm{sw}_y(\mathcal{F})$ in terms of the *break decomposition* of the G-module V induced by the filtration (G^γ) of the group G_x,

$$V = \bigoplus_{\gamma\in\Gamma_x\otimes\mathbb{Q}} V(\gamma).$$

Here $V(\gamma) \subset V$ is defined for $\gamma < 1$ as the subset of $\langle v - \sigma(v) \mid v \in V, \sigma \in G^\gamma\rangle$ consisting of elements which are invariant under $\cup_{\delta<\gamma} G^\delta$. Furthermore, $V(1) := V^P$, where $P = \cup_{\gamma<1} G^\gamma$ is the maximal p-subgroup of G_x. An element $\gamma \in \Gamma_x \otimes \mathbb{Q}$ is called a *break* for ρ if $V(\gamma) \neq 1$. It is clear that a break is either a jump or equal to 1. By [13], Corollary 8.4 we have

$$\mathrm{sw}_y(\mathcal{F}) \;=\; \sum_{i=1}^{l} \#\gamma_i \cdot \dim V(\gamma_i). \tag{4}$$

The definition of $\delta(\mathcal{F})$ is analogous. Let $\log_q : \mathbb{R}_{>0} \to \mathbb{R}$ be the logarithm to the basis $q := |\varpi^{-1}|$, where ϖ is a uniformizer of the discretely valued field K_0. We set

$$\delta_x(\sigma) := -|G_x| \log_q h_x^\flat(\sigma) \quad \text{if } \sigma \neq 1, \quad \delta_x(1) := -\sum_{\sigma \neq 1} \delta(\sigma)$$

and

$$\delta_y := \mathrm{Ind}_{G_x}^G \delta_x.$$

Finally, we define the discriminant conductor of $\mathcal{F}$ at y as follows:

$$\delta_y(\mathcal{F}) := \langle \delta_y, \tau \rangle_G.$$

Similar to (4), we have a formula which computes $\delta_y(\mathcal{F})$ in terms of the break decomposition of the G-module V:

$$\delta_y(\mathcal{F}) = \sum_{i=1}^{l} -\log_q \gamma_i^\flat \cdot \dim V(\gamma_i). \tag{5}$$

In particular, $\delta_y(\mathcal{F})$ is a nonnegative rational number.

2.6 Comparison with Huber's and Ramero's theory Let X be an open analytic curve with field of definition K. A *compactification* of X is an open embedding $\mathsf{X} \subset \mathsf{X}_1$, where X_1 is an open analytic curve over K such that X is contained in an affinoid subdomain of X_1. Let $\mathsf{X}_1^{\mathrm{ad}}$ denote the analytic adic space associated to X_1, see [12]. We will associate to every end $x \in \partial\mathsf{X}$ a certain point $x^{\mathrm{ad}} \in \mathsf{X}_1^{\mathrm{ad}}$.

Let $\mathsf{A} \subset \mathsf{X}$ be an open annulus representing the end x. After enlarging the field K, if necessary, there exists a formal model $\mathcal{X}$ of X_1 with reduced special fiber $\mathcal{X}_s$, a closed subset $Z \subset \mathcal{X}_s$ and a closed point $z \in Z$ such that $\mathsf{X} =]Z[_{\mathcal{X}}$ and $\mathsf{A} =]z[_{\mathcal{X}}$. Since A is an open annulus, z is an ordinary double point of $\mathcal{X}_s$. One of the irreducible components of $\mathcal{X}_s$ passing through z, say W, is not contained in Z; otherwise, A would be contained in an affinoid which is itself contained in X, contradicting Definition 2.2. Let $\mathrm{Spm} A_K \subset \mathsf{X}_1$ be an affinoid subdomain containing X. The component W gives rise to a discrete valuation $|\cdot|_\eta^{\mathrm{ad}}$ on the fraction field of the affinoid algebra A_K. The residue field of this valuation can be identified with $k(W)$, the function field of W. By the definition of adic spaces, $|\cdot|_\eta^{\mathrm{ad}}$ corresponds to a point $\eta \in \mathsf{X}_1^{\mathrm{ad}}$. In terms of the classification of points of $\mathsf{X}_1^{\mathrm{ad}}$ in [13], §5, η is a point of *type II*. Such a point has infinitely many proper specialization to points of *type III*. In particular, let $|\cdot|_x^{\mathrm{ad}}$ denote the rank two valuation on the fraction field of A_K which is the composition of the valuation $|\cdot|_\eta^{\mathrm{ad}}$ with the valuation on $k(W)$ corresponding to the point $z \in W$. Then $|\cdot|_x^{\mathrm{ad}}$ corresponds to a point $x^{\mathrm{ad}} \in \mathsf{X}_1^{\mathrm{ad}}$ which is of type III. By the definition of $\mathsf{X}_1^{\mathrm{ad}}$ as a topological space, x^{ad} is contained in the closure of the point η. Extend the valuation $|\cdot|_\eta^{\mathrm{ad}}$ to $\mathrm{Frac}(A_K \otimes_K K^{\mathrm{ac}})$ in the obvious way, and let K_x^{ad} denote the henselization.

Proposition 2.6 *There is a canonical injection $K_x^{\mathrm{ad}} \hookrightarrow K_x$ of valued fields which induces an isomorphism on the value groups of the valuations.*

Proof: Let $\mathcal{U} = \mathrm{Spf} B$ be an affine open formal subscheme of $\mathcal{X}$ containing the point z. The inclusion $\mathcal{U}\otimes K \hookrightarrow \mathsf{X}_1$ corresponds to a morphism $A_K \to B\otimes K$. On the other hand, the formal completion of $\mathcal{U}$ in z can be identified with the minimal formal model of the annulus A. In particular, $\mathcal{O}_{\mathsf{A}}^\circ = \hat{\mathcal{O}}_{\mathcal{U},z}$. Hence we obtain a field extension

$$\mathrm{Frac}(A_K \otimes_K K^{\mathrm{ac}}) \to \mathrm{Frac}(\mathcal{O}_{\mathsf{A}}^\circ \otimes_R K^{\mathrm{ac}}).$$

By construction this is an extension of valued fields inducing an isomorphism of the valuation groups, with respect to the valuations $|\cdot|_x^{\mathrm{ad}}$ on the left and the valuation $|\cdot|_x$ on the right. The field K_x^{ad} is defined as the henselization of field on the left, whereas the field K_x is the completion of the field on the right. Since K_x is henselian, we obtain the desired injection $K_x^{\mathrm{ad}} \hookrightarrow K_x$. $\square$

Let $\mathcal{F}$ be an admissible sheaf on X and suppose that $\mathcal{F}$ extends to an admissible sheaf $\mathcal{F}_1$ on X_1. Let $x \in \partial\mathsf{X}$ be an end and $x^{\mathrm{ad}} \in \mathsf{X}_1^{\mathrm{ad}}$ the corresponding adic point on X_1. Let $\eta \in \mathsf{X}_1^{\mathrm{ad}}$ be the unique generalization of x^{ad}. According to [13] and [18], we can define a Swan conductor and a discriminant conductor

$$\mathrm{sw}_{x^{\mathrm{ad}}}(\mathcal{F}_1) \in \mathbb{Z}, \qquad \delta_\eta(\mathcal{F}_1) \in \mathbb{R}.$$

These are defined in the same manner as $\mathrm{sw}_x(\mathcal{F})$ and $\delta_x(\mathcal{F})$, with the valued field K_x replaced by K_x^{ad}. Therefore, Proposition 2.6 shows:

Corollary 2.7 *For every compactification $\mathsf{X}_1 \supset \mathsf{X}$ and every extension $\mathcal{F}_1$ of $\mathcal{F}$ to X_1 we have*

$$\mathrm{sw}_x(\mathcal{F}) = \mathrm{sw}_{x^{\mathrm{ad}}}(\mathcal{F}_1), \qquad \delta_x(\mathcal{F}) = \delta_\eta(\mathcal{F}_1).$$

Remark 2.8 It is plausible that for every admissible sheaf $\mathcal{F}$ on X there exists a compactification $\mathsf{X}_1 \supset \mathsf{X}$ and an extension $\mathcal{F}_1$ of $\mathcal{F}$ to X_1, but the author does not know how to prove this.

2.7 Continuity Fix an element $R \in \Gamma \cup \{0\}$, $R < 1$. If $R \neq 0$ we set

$$\mathsf{X} := \mathsf{C}(R,1) = \{\, t \mid R < |t| < 1 \,\},$$

which is an open annulus; if $R = 0$ we let

$$\mathsf{X} := \mathsf{D}(0,1) = \{\, t \mid |t| < 1 \,\}$$

be the standard open disk. For every $r \in \Gamma$ with $R < r \leq 1$ we set $s := -\log_q r$ and define an open subset

$$\mathsf{X}_s := \{\, t \in \mathsf{X}_s \mid |t| < r \,\},$$

which is again an open annulus or an open disk. We let $x_s \in \partial \mathsf{X}_s$ denote the 'exterior' end corresponding to the family of annuli $C(r', r)$ for $R < r' < r$. If $R \neq 0$ we shall identify the 'interior' end of X_s for all s and denote it by y.

Let $\mathcal{F}$ be an admissible sheaf on X. For $s = -\log_q r$ as above we define

$$\delta_{\mathcal{F}}(s) := \delta_{x_s}(\mathcal{F}|_{\mathsf{X}_s}), \qquad \mathrm{sw}_{\mathcal{F}}(s) := \mathrm{sw}_{x_s}(\mathcal{F}|_{\mathsf{X}_s}).$$

Proposition 2.9 *The association $s \mapsto \delta_{\mathcal{F}}(s)$ extends to a continuous and piecewise linear function $\delta_{\mathcal{F}} : [0, -\log_q R] \to \mathbb{R}_{\geq 0}$. Similarly, the association $s \mapsto \mathrm{sw}_{\mathcal{F}}(s)$ extends to a right continuous and piecewise constant function $\mathrm{sw}_{\mathcal{F}} : [0, -\log_q R] \to \mathbb{Z}$. Furthermore:*

(i) *The function $\delta_{\mathcal{F}}$ is convex.*

(ii) *The function $\mathrm{sw}_{\mathcal{F}}$ is decreasing.*

(iii) *For all $s \in [0, -\log_q R)$ we have*

$$\frac{\partial}{\partial s}\delta_{\mathcal{F}}(s+) = -\mathrm{sw}_{\mathcal{F}}(s).$$

(iv) *If $R = 0$ then δ is decreasing and eventually zero.*

Proof: If $s > 0$ then X is a compactification of X_s and hence the end $x_s \in \partial \mathsf{X}_s$ corresponds to an adic point $x^{\mathrm{ad}} \in \mathsf{X}^{\mathrm{ad}}$. Using Corollary 2.7, one can therefore deduce Proposition 2.9 for $s > 0$ from results of [18], in particular Theorem 2.3.35 and Proposition 3.3.26. At $s = 0$ we do not have a definition of $\delta_{\mathcal{F}}$ and $\mathrm{sw}_{\mathcal{F}}$ in terms of the adic space X^{ad}, and we cannot directly use the results of [18]. But all that remains to be shown is that $\delta_{\mathcal{F}}$ and $\mathrm{sw}_{\mathcal{F}}$ are right continuous at $s = 0$.

Let $f : \mathsf{Y} \to \mathsf{X}$ be an étale G-Galois cover such that $f^*\mathcal{F}$ is constant; then $\mathcal{F} \cong (f_*\bar{\mathbb{Q}}_\ell)[\tau]$ for some representation τ of G. For $s \in (0, -\log_q R)$ and $r := q^{-s}$ set $A_s := \{t \in \mathsf{X} \mid |t| < r\}$ and let $\mathsf{B} \subset \mathsf{Y}$ be a connected component of $f^{-1}(\mathsf{A})$. Using the semistable reduction theorem, one easily shows that B is an open annulus for all s sufficiently close to zero (see [21]). Therefore, the following lemma implies that the function $\delta_{\mathcal{F}}$ (resp. $\mathrm{sw}_{\mathcal{F}}$) is linear (resp. constant) on the interval $[0, s]$. The proof of the proposition is now complete. $\square$

Lemma 2.10 *Let $\mathsf{X} = \mathsf{C}(R, 1)$ be an open annulus and $\mathsf{X}' = \mathsf{C}(R, r)$ a subannulus, with $R < r < 1$. Let $x \in \partial \mathsf{X}$ be the 'exterior' end corresponding to the family of annuli $C(r', 1)$ for $R < r' < 1$. Likewise, let $x' \in \partial \mathsf{X}'$ be the 'exterior' end of X'. Let G be a finite group acting faithfully on X and fixing the end x. Then the following holds.*

(i) *The action of G fixes the annulus X' and the end x'.*

(ii) *For all $\sigma \in G$, $\sigma \neq 1$ we have*

$$\#h_{x'}(\sigma) = \#h_x(\sigma)$$

and

$$\log_q h^\flat_{x'}(\sigma) = \log_q h^\flat_x(\sigma) + \#h_x(\sigma) \cdot \log_q r.$$

Proof: This follows from a direct computation, using Example 2.4 and (2). $\square$

Remark 2.11 The last part of its proof, including Lemma 2.10, shows that most of Proposition 2.9 (except maybe for the convexity of $\delta_{\mathcal{F}}$) is a rather straightforward consequence of the Semistable Reduction Theorem. This argument essentially goes back to Lütkebohmert's paper [17] on the non-archimedian Riemann Existence Theorem. See also [19].

3 The boundary of Lubin-Tate space

In this section we compute the filtration by higher ramification groups at the boundary of the étale cover $f_n : \mathsf{X}(\varpi^n) \to \mathsf{X}(1)$ of Lubin-Tate spaces of dimension one.

3.1 Notation Let F be a non-archimedian local field, e.g. a field which is complete with respect to a discrete valuation $|\cdot|$ and whose residue field is finite, say with $q = p^f$ elements. We let $\mathcal{O}$ denote the ring of integers of F. Furthermore, we choose a uniformizer ϖ of $\mathcal{O}$. We assume that $|\varpi| = q^{-1}$.

Let $K_0 := \hat{F}^{\rm nr}$ denote the completion of the maximal unramified extension of F, R_0 the valuation ring of K_0 and k the residue field of K_0. We choose an algebraic closure $K^{\rm ac}$ of K_0 and extend the valuation $|\cdot|$ from F to $K^{\rm ac}$. The valuation group of $|\cdot|$ is denoted by Γ. We write K to denote a finite extension of K_0 contained in $K^{\rm ac}$ and R for the valuation ring of K. Note that all this notation is consistent with the notation used in Section 1.

Let Σ_0 be the unique formal $\mathcal{O}$-module of height two over k. Let $\mathcal{X}(1) = \mathrm{Spf} A_0$ be the universal deformation space of Σ_0, and let $\Sigma^{\rm uv}/A_0$ denote the universal deformation of Σ_0. For each integer $n \geq 0$ be denote by $\mathcal{X}(\varpi^n) = \mathrm{Spf} A_n$ the universal deformation space parameterizing deformations of Σ_0 with a Drinfeld level structure of level ϖ^n. We denote by

$$\phi_n : (\mathcal{O}\varpi^{-n}/\mathcal{O})^2 \to (\Sigma^{\rm uv} \otimes_{A_0} A_n)[\varpi^n]$$

the tautological level-ϖ^n-structure on $\Sigma^{\rm uv} \otimes_{A_0} A_n$. For each $n \geq 0$ we let

$$\mathsf{X}(\varpi^n) := \mathcal{X}(\varpi^n) \otimes_{R_0} K$$

be the generic fiber of the formal scheme $\mathcal{X}(\varpi^n)$. We regard $\mathsf{X}(\varpi^n)$ as an open analytic curve; recall that this means essentially that the field of definition K is a 'sufficiently large' finite extension of K_0. We remark that the right choice of K will in general depend on n. For instance, we want that the connected components of $\mathsf{X}(\varpi^n)$ stay connected over any extension of K. For this it is

necessary and sufficient that the field K contains the abelian extension of K_0 corresponding to the group of n-units $1+\mathcal{O}\varpi^n \subset \mathcal{O}^\times$ via local class field theory.

Write G_n for the finite group $\mathrm{GL}_2(\mathcal{O}/\varpi^n)$. It is well known that the natural map

$$f_n : \mathsf{X}(\varpi^n) \to \mathsf{X}(1)$$

is an étale G_n-torsor. An element $\sigma \in G_n$ acts on $\mathsf{X}(\varpi^n)$ by composing the tautological level-ϖ^n-structure ϕ_n with σ, considered as a linear automorphism σ of $(\mathcal{O}\varpi^{-n}/\mathcal{O})^2$.

3.2 The ramification filtration By the fundamental result of Lubin-Tate and Drinfeld, the universal deformation ring of Σ_0 is a power series ring over R_0, i.e. $A_0 = R_0[[T]]$. This means that we can identify the Lubin-Tate space $\mathsf{X}(1)$ with the standard open unit disk.

Let $x \in \partial\mathsf{X}(1)$ be the unique end of $\mathsf{X}(1)$. Fix an integer $n \geq 1$ and choose an end $y \in f_n^{-1}(x) = \partial\mathsf{X}(\varpi^n)$. Let K_y/K_x denote the corresponding extension of valued fields and $G_y \subset G_n$ its Galois group. Let $h_0 < \cdots < h_l$ denote the jumps in the ramification filtration $(G_h)_h$ of the group G_y. (See Section 1 for the notation.) As a convenient notational device, we define the group homomorphism

$$-\mathrm{Log} : \Gamma_y \to \mathbb{Q} \times \mathbb{Z}, \quad (r,\, \gamma_y^i) \mapsto (-\log_q r,\, i).$$

Proposition 3.1 *We may choose $y \in f_n^{-1}(x)$ such that*

$$G_y = \Big\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathcal{O}_{F,n}^\times,\, b \in \mathcal{O}_{F,n} \Big\}.$$

Then the jumps and the corresponding higher ramification groups are as follows.

(i) *For $i = 1, \ldots, n-1$ we have*

$$-\mathrm{Log}\, h_i = (0,\, q^{2i} - 1)$$

and

$$G_{h_i} = \Big\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \equiv 1 \pmod{\varpi^i} \Big\}.$$

(ii) *For $j = 0, \ldots, n-1$ and $i = n + j$ we have*

$$-\mathrm{Log}\, h_i = \big(\frac{1}{q^{n-j-1}(q-1)},\, -q^{2n-1} - 1\big)$$

and

$$G_{h_i} = \Big\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \equiv 0 \pmod{\varpi^j} \Big\}.$$

A proof of this proposition will be given in §3.3 below.

Corollary 3.2 *For the jumps $\gamma_i = (\gamma_i^\flat, \#\gamma_i)$ in the upper numbering we have the following formulas:*

$$\#\gamma_i = \begin{cases} (q+1)(1+q+\ldots+q^{i-1}), & i = 1,\ldots,n-1 \\ -\dfrac{q+1}{q-1}, & i = n,\ldots,2n-1, \end{cases} \tag{6}$$

and

$$-\log_q \gamma_{2n-1}^\flat = \frac{nq-n+1}{q-1}. \tag{7}$$

Proof: This follows from Proposition 3.1 by a direct computation, using (3). □

3.3 The proof of Proposition 3.1 In this section we write $\Sigma = \Sigma^{\mathrm{uv}}$ for the universal deformation of the formal $\mathcal{O}$-module Σ_0 over the ring $A_0 = R_0[[T]]$. We fix a parameter X for the formal group law underlying Σ. We write $[a](X) = aX + \ldots \in A_0[[X]]$ for the endomorphisms of Σ corresponding to an element $a \in \mathcal{O}$. It is well known that for a suitable choice of the parameters X and T we have

$$[\varpi](X) \equiv \varpi X + TX^q + X^{q^2} \pmod{\varpi X^q}. \tag{8}$$

See e.g. [11]. For $n \geq 1$, let

$$\phi_n : \mathcal{O}_n^2 \longrightarrow \Sigma[\varpi^n]$$

denote the tautological Drinfeld level-ϖ^n-structure over $\mathcal{X}(\varpi^n) = \mathrm{Spf} A_n$. We identify the $\mathcal{O}$-module $\Sigma[\varpi^n]$ with the subset of the maximal ideal of A_n whose elements satisfy the equation

$$[\varpi^n](X) = \underbrace{[\varpi] \circ \cdots \circ [\varpi]}_{n \text{ times}}(X) = 0. \tag{9}$$

The $\mathcal{O}$-module structure on this set is given by the formal group law $X +_\Sigma Y = X + Y + \ldots \in A[[X,Y]]$ and the formal power series $[a](X) = aX + \ldots$. It is shown in [8] that

$$A_n = A[u_n, v_n], \tag{10}$$

where

$$u_n := \phi_n(1,0), \quad v_n := \phi_n(0,1)$$

is the standard $\mathcal{O}_n$-basis of $\Sigma[\varpi^n]$ determined by ϕ_n. An element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_n)$ acts on A_n over A by the formula

$$\begin{aligned} u_n &\longmapsto [a](u_n) +_\Sigma [c](v_n), \\ v_n &\longmapsto [b](u_n) +_\Sigma [d](v_n). \end{aligned} \tag{11}$$

Let $H_n \subset \Sigma[\varpi^n]$ be the sub-$\mathcal{O}$-module generated by u_n. Let $\Sigma'_n := \Sigma/H_n$ be the quotient (in the category of formal $\mathcal{O}$-modules over A_n) of Σ under H_n. With respect to a suitable parameter X', the canonical morphism $\Sigma \to \Sigma'_n$ is given by $X' = \alpha_n(X) \in A_n[[X]]$, where

$$\alpha_n(X) \;=\; \prod_{a\in\mathcal{O}_n} (X +_\Sigma [a](u_n)), \tag{12}$$

see e.g. [11]. Let $\beta_n : \Sigma'_n \to \Sigma$ be the 'dual' isogeny to α_n, i.e. the formal power series $\beta_n \in A_n[[X']]$ such that

$$[\varpi^n](X) \;=\; \beta_n \circ \alpha_n(X). \tag{13}$$

Let $x \in \partial\mathsf{X}(1)$ be the unique end of the disk $\mathsf{X}(1)$. An end $y \in f_n^{-1}(x)$ corresponds to a valuation $|\cdot|_y$ on $\mathrm{Frac}(A_n)$ extending $|\cdot|_x$. It follows from (10) that the field K_y is the finite extension of K_x generated by u_n, v_n,

$$K_y = K_x(u_n, v_n). \tag{14}$$

We write $\mathrm{val}_y := -\log_q|\cdot|_y^\flat : K_y^\times \to \mathbb{Q}$ for the exponential version of the rank one valuation $|\cdot|_y^\flat$.

Lemma 3.3 *We can choose $y \in f_n^{-1}(x)$ such that*

$$\mathrm{val}_y(u_n) \;=\; \frac{1}{(q-1)q^{n-1}}, \quad \mathrm{val}_y(v_n) \;=\; 0.$$

If this is the case then G_y is contained in the subgroup of $\mathrm{SL}_2(\mathcal{O}_n)$ consisting of upper triangular matrices.

Proof: (Compare with [16].) We prove the lemma by induction on n. Suppose first that $n = 1$. By (8), the Newton polygon of $[\varpi](X)$ with respect to val_x has a unique negative slope $-1/(q-1)$ over the interval $[1,\ldots,q]$. It follows that $\Sigma[\varpi]$ has a 'canonical subgroup' H whose nonzero elements w satisfy $\mathrm{val}_y(w) = 1/(q-1)$. For $w \in \Sigma[\varpi] - H$ we have $\mathrm{val}_y(w) = 0$. By classical valuation theory, we can choose $y = y_1 \in \Psi_1^{-1}(x)$ such that $H = H_1 = \langle u_n\rangle$. Then G_y is contained in the stabilizer of H_1 which consists of upper triangular matrices (use (11)). This proves the case $n = 1$. The induction step from n to $n+1$ is similar. For instance, one uses the fact that u_{n+1} is a solution of the equation

$$[\varpi](X) - u_n = 0.$$

By (8) and the induction hypothesis, the first slope of the Newton polygon of this equation with respect to val_{y_n} is $-1/q^n(q-1)$. □

From now on we shall assume that y is chosen as in Lemma 3.3. Set $\mathfrak{p}_y$ denote the prime ideal of the valuation ring K_y^+ of K_y corresponding to the valuation $|\cdot|_y^\flat$. We write $K_y^\sim := \mathrm{Frac}(K_y^+/\mathfrak{p}_y)$ for its residue field. Lemma 3.3 and Equation (12) show that we have the congruence

$$\alpha_n(X) \;\equiv\; X^{q^n} \pmod{\mathfrak{p}_y}. \tag{15}$$

Using (8), (13) and induction on n one concludes that

$$\beta_n(X) \equiv E \circ E^{(1)} \circ \cdots \circ E^{(n-1)}(X) \pmod{\mathfrak{p}_y}, \tag{16}$$

where

$$E^{(i)}(X) := T^{q^i} X + X^q.$$

Set $w_n := \alpha_n(v_n)$. Then $\beta_n(w_n) = 0$ and by (15) we have $w_n \equiv v_n^{q^n} \pmod{\mathfrak{p}_y}$. In particular, we have $\mathrm{val}_y(w_n) = 0$. Let $M_n := K_x(w_n)$ denote the field extension of K_x generated by w_n. We write $M_n^\sim := \mathrm{Frac}(M_n^+/(\mathfrak{p}_y \cap M_n^+))$ for the residue field of the valuation $\mathrm{val}_y|_{M_n}$. Note that the restriction of $\#_y$ to M_n induces a discrete valuation on $M_n^\sim$. Let $U_n \subset \mathrm{SL}_2(\mathcal{O}_n)$ denote the subgroup of upper triangular, unipotent matrices.

Lemma 3.4 (i) *The image of w_n in the residue field $M_n^\sim$ is a uniformizer (with respect to the discrete valuation induced by $\#_y$).*

(ii) *The field M_n is the fixed field of K_y of the subgroup $U_n \cap G_y \subset G_y$.*

(iii) *The map $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \mapsto a$ induces an isomorphism*

$$G_y/(G_y \cap U_n) \cong \mathcal{O}_n^\times.$$

Proof: Let z_n denote the image of w_n in $M_n^\sim$. By induction, we define elements $z_m \in M_n^\sim$ for $m = n-1, \ldots, 0$ by

$$z_{m-1} := E^{(m-1)}(z_m).$$

By (16) we have $z_0 \equiv \beta_n(w_n) \equiv 0 \pmod{\mathfrak{p}_y}$. Let $L_m := K_x^\sim(z_m) \subset M_n^\sim$ be the subextension of residue fields generated by z_m. The element z_1 satisfies the equation over $K_x^\sim$

$$E(X)/X = X^{q-1} + T = 0.$$

Since T is a uniformizer of $K_x^\sim$, we have $[L_1 : K_x^\sim] = q-1$ and z_1 is a uniformizer of L_1. For $m = 1, \ldots, n-1$, the element z_{m+1} is a solution of the equation over L_m

$$E^{(m)}(X) - z_m = X^q + T^{q^m} X - z_m = 0. \tag{17}$$

By induction, one proves that this is an Eisenstein equation and hence irreducible over L_m and that z_{m+1} is a uniformizer of L_{m+1}. For $m = n-1$, this gives Part (i) of the lemma. It also follows that

$$[M_n : K_x] = [M_n^\sim : K_x^\sim] = (q-1)q^{n-1} = |\mathcal{O}_n^\times|. \tag{18}$$

By the definition of w_n and the isogeny α_n, the subfield $M_n \subset K_y$ is fixed under the action of $G_y \cap U_n$. On the other hand, we have an injective homomorphism $G_y/(G_y \cap U_n) \hookrightarrow \mathcal{O}_n^\times$. Now (18) implies Part (ii) and (iii) of the lemma. $\Box$

Lemma 3.5 (i) *We have*

$$[K_y : M_n] = [K_y^\sim : M_n^\sim] = q^n = |\mathcal{O}_n|.$$

(ii) *The image of v_n in $K_y^\sim$ is a uniformizer (with respect to the discrete valuation induced by $\#_y$).*

(iii) *The Galois group $G_y = \mathrm{Gal}(K_y/K_x)$ consists of all upper triangular matrices in $\mathrm{SL}_2(\mathcal{O}_n)$.*

Proof: By (15) and the definition of w_n we have

$$v_n^{q^n} \equiv w_n \pmod{\mathfrak{p}_y}.$$

Together with Lemma 3.4 (i) it follows that (a) the field extension $M_x(v_n)/M_x$ has degree q^n (and its residue field extension $M_x(v_n)^\sim/M_n^\sim$ is purely inseparable) and (b) v_n is a uniformizer of $M_n(v_n)^\sim$. By Lemma 3.4 (ii) we have $[K_y : M_n] = |G_y \cap U_n| \leq |U_n| = q^n$. We conclude from (a) that $K_y = M_n(v_n)$ and hence that Part (i) of the lemma holds. Also, Part (ii) follows from (b) above. Finally, we have shown that $U_n \subset G_y$, and so Lemma 3.4 (iii) implies Part (iii) of the lemma. □

Lemma 3.6 *For $m = 1, \ldots, n$ we have*

$$\mathrm{val}_y(u_m) = \frac{1}{(q-1)q^{m-1}}, \qquad \#_y u_m = -q^{2n-1} \tag{19}$$

$$\mathrm{val}_y(v_n) = 0, \qquad \#_y v_m = q^{2(n-m)}. \tag{20}$$

Proof: Let y_m denote the image of y in $\Psi_m^{-1}(x)$. Then $\mathrm{val}_y|_{K_{y_m}} = \mathrm{val}_{y_m}$. Therefore, the formulae for $\mathrm{val}_y(u_m)$ and $\mathrm{val}_y(v_m)$ follow from Lemma 3.3 and the choice of y. We also have

$$\#_y|_{K_{y_m}} = [K_y : K_{y_m}] \cdot \#_{y_m} = q^{2(m-n)} \cdot \#_{y_m}, \tag{21}$$

by Lemma 3.5 (iii). Therefore, the formula for $\#_y(v_m)$ follows from Lemma 3.5 (ii). Moreover, in order to prove the formula for $\#_y(u_m)$ it suffices to show that $\#_y(u_n) = -q^{2n-1}$.

We proceed by induction on n. For $n = 1$ we note that u_1 satisfies the equation $[\varpi](X) = 0$ over K_x. Substituting $X = \varpi^{1/(q-1)}Y$ and using (8), we see that the image of $\varpi^{-1/(q-1)}u_1$ in $K_x^\sim$ is a solution to the equation

$$Y + TY^q = 0.$$

We conclude that

$$q \cdot \#_{y_1} u_1 + \#_{y_1} T = \#_{y_1} u_1,$$

which implies $\#_{y_1} u_1 = -q$ and proves the claim for $n = 1$. The induction step from n to $n+1$ is again similar: substitute $X = \varpi^{1/(q-1)q^{n-1}}Y$ into the equation $[\varpi](X) = u_n$ (of which u_{n+1} is a solution). □

We can now finish the proof of Proposition 3.1. If we choose y as in Lemma 3.3 then the first claim of the proposition (which determines the group G_y) is

proved by Lemma 3.5 (iii). Let

$$\sigma \;=\; \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix}$$

be an arbitrary element of $G_y - \{1\}$. To prove the proposition, it will suffices to compute $h_y(\sigma) \in \Gamma_y$. If $a \neq 1 \in \mathcal{O}_n$ then we set $i := \mathrm{val}_\varpi(a-1) \in \{0, \ldots, n-1\}$. If $a = 1 \in \mathcal{O}_n$ then we set $j := \mathrm{val}_\varpi(b)$ and $i := n + j$. We claim that

$$-\mathrm{Log}\, h_y(\sigma) \;=\; \begin{cases} (0,\, 0), & \text{for } i = 0, \\ (0,\, q^{2i} - 1), & \text{for } i = 1, \ldots, n-1, \\ (\dfrac{1}{(q-1)q^{n-j-1}},\, -q^{2n-1} - 1), & \text{for } i \geq n. \end{cases} \tag{22}$$

Clearly, this claim implies the proposition.

By (2), Lemma 3.5 (ii) and (11), we have

$$h_y(\sigma) \;=\; \Big|\frac{\sigma(v_n) - v_n}{v_n}\Big|_y \;=\; \Big|\frac{([b](u_n) +_\Sigma [a](v_n)) - v_n}{v_n}\Big|_y. \tag{23}$$

If $a \not\equiv 1 \pmod{\varpi}$ then $[a](v_n) = av_n + (\text{higher terms}) \equiv av_n \not\equiv 0 \pmod{\mathfrak{p}_y}$ and $[b](u_n) \equiv 0 \pmod{\mathfrak{p}_y}$, by (19) and (20). From (23) we conclude that

$$-\mathrm{Log}\, h_y(\sigma) = -\mathrm{Log}\, |a|_y = (0,\, 0).$$

This proves (22) for $i = 0$. Suppose now that $i \in \{1, \ldots, n-1\}$ and write $a = 1 + \varpi^i c$. Then

$$[a](v_n) = [1 + c\varpi^i](v_n) = v_n +_\Sigma [c](v_{n-i}) = v_n + cv_{n-i} + \ldots,$$

where the remaining terms are units in $(K_y^+)_{\mathfrak{p}_y}$ whose image in $K_y^\sim$ have valuation $> \#_y v_{n-i} = q^{2i}$, by (20). It follows that

$$-\mathrm{Log}\, h_y(\sigma) = -\mathrm{Log}\,\Big|\frac{[a](v_n) - v_n}{v_n}\Big| = -\mathrm{Log}\,\Big|\frac{cv_{n-i}}{v_n}\Big| = (0,\, q^{2i} - 1).$$

This proves (22) for $i = 1, \ldots, n-1$. Finally, for $i \geq n$ we write $b = \varpi^j c$ and get

$$\sigma(v_n) = v_n +_\Sigma [c](u_{n-j}) = v_n + cu_{n-j} + \ldots,$$

where the remaining terms have $\mathfrak{p}_y$-valuation $> \mathrm{val}_y(u_{n-j}) = 1/(q-1)q^{n-j-1}$. Using (19) we conclude that

$$-\mathrm{Log}\, h_y(\sigma) = -\mathrm{Log}\,\Big|\frac{cu_{n-j}}{v_n}\Big| = \big(\frac{1}{(q-1)q^{n-j-1}},\, -q^{2n-1} - 1\big).$$

This completes the proof of Proposition 3.1. $\square$

4 Ramification of supercuspidal representations

In this section we apply the results of the previous section to compute the Swan and the discriminant conductor of the sheaves on the Lubin-Tate space corresponding to the types of supercuspidal representations of $\mathrm{GL}_2(F)$. We also draw some conclusions concerning the cohomology of the Lubin-Tate tower.

4.1 The cohomology of the Lubin-Tate tower We continue with the notation introduced in the last section. We also choose a prime number ℓ which is strictly bigger than p. Then the order of the finite groups $G^{(n)} = \mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^n)$ are all prime to ℓ. We write $K := \mathrm{GL}_2(\mathcal{O})$ and $G := \mathrm{GL}_2(F)$. Let W_F denote the Weil group of F and $I_F \subset W_F$ the inertia subgroup.

Fix $n \geq 0$ and $i \in \{0, 1, 2\}$. We let $H^i(\mathsf{X}(\varpi^n), \bar{\mathbb{Q}}_\ell)$ denote étale cohomology of the rigid analytic space $\mathsf{X}(\varpi^n) \otimes_{K_0} \hat{K}^{\mathrm{ac}}$, in the sense of Berkovich [1]. See also [21], §2.1. Similarly, one has cohomology with compact support, $H^i_c(\mathsf{X}(\varpi^n), \bar{\mathbb{Q}}_\ell)$. We define $H^1_{\mathrm{p}}(\mathsf{X}(\varpi^n), \bar{\mathbb{Q}}_\ell)$, the *parabolic cohomology* of $\mathsf{X}(\varpi^n)$, as the image of the natural map $H^1_c(\mathsf{X}(\varpi^n), \bar{\mathbb{Q}}_\ell) \to H^1(\mathsf{X}(\varpi^n), \bar{\mathbb{Q}}_\ell)$. This is a finite dimensional $\bar{\mathbb{Q}}_\ell$-vectorspace, together with a continuous action of the group $G^{(n)} \times \mathcal{O}_B^\times \times I_F$.

Set

$$\mathcal{H}_0 := \varinjlim_n H^1_{\mathrm{p}}(\mathsf{X}(\varpi^n), \bar{\mathbb{Q}}_\ell).$$

This is an infinite-dimensional vector space with a continuous action of the group $K \times \mathcal{O}_B^\times \times I_F$. This action extends, in a natural way, to an action of a certain subgroup

$$(G \times B^\times \times W_F)_0 \subset G \times B^\times \times W_F.$$

This subgroup is the kernel of the homomorphism $G \times B^\times \times W_F \to \mathbb{Z}$ which sends (g, b, σ) to the normalized valuation of $\det(g)^{-1} N(b) \mathrm{cl}(\sigma)$. Here $N : B^\times \to F^\times$ is the reduced norm and $\mathrm{cl} : W_F \to F^\times$ the inverse reciprocity map. See e.g. [9] or [20]. We let $\mathcal{H}$ denote the representation of $G \times B^\times \times W_F$ induced from $\mathcal{H}_0$.

Theorem 4.1 (Carayol) (i) *Let π be an irreducible supercuspidal representation of G over the field $\bar{\mathbb{Q}}_\ell$. As a representation of $B^\times \times W_F$ we have*

$$\mathrm{Hom}_G(\pi, \mathcal{H}) \cong \mathrm{JL}(\pi)^\vee \otimes \mathrm{L}(\pi)',$$

where $\mathrm{JL}(\pi)$ is the image of π under the local Jacquet-Langlands correspondence and $\mathrm{L}(\pi)$ is the image under the Hecke-correspondence (a certain normalization of the local Langlands correspondence).

(ii) *If π is a smooth admissible irreducible representation of G which is not supercuspidal then $\mathrm{Hom}_G(\pi, \mathcal{H}) = 0$.*

Proof: Part (a) of this theorem is proved in [4], see also [5]. Part (b) is certainly known to the experts, but there seems to be no explicit reference in the literature. We shall deduce Part (b) from our computations of Swan conductors, see Corollary 4.10. □

Carayol has conjectured [5] that Theorem 4.1 extends to the group $\mathrm{GL}_n(F)$ and the corresponding Lubin-Tate spaces of dimension $n-1$ for all $n \geq 2$. This conjecture has been proved by Harris and Taylor [10], along with the local Langlands correspondence for GL_n. Their method is a generalization of Carayol's method [4], which in turn generalizes arguments of Deligne [7]. These arguments are quite indirect.

In a work in preparation, the author intends to give a new and more direct proof of Theorem 4.1 which relies on an analysis of the stable reduction of the spaces $\mathsf{X}(\varpi^n)$, studied in [20]. The results of this paper, in particular Theorem 4.3 and Theorem 4.5, are a crucial ingredient for this analysis.

4.2 The type of a supercuspidal representation

In the following, all representations are defined over the field $\bar{\mathbb{Q}}_\ell$. Recall that we are concerned with the groups $G := \mathrm{GL}_2(F)$, $K := \mathrm{GL}_2(\mathcal{O})$ and $G^{(n)} = \mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^n)$. We let $K_n \subset K$ denote the principal congruence subgroup modulo $\wp^n$. Let $U = U_0 \subset K$ and $U_n \subset K_n$ be the subgroups containing the upper triangular, unipotent matrices. We write $H^{(n)} \subset G^{(n)}$ for the image of a subgroup $H \subset K$. Let

$$K' := \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \;\mid\; c \equiv 0 \mod \wp \Big\}$$

denote the Iwahori subgroup of K and set, for $n \geq 1$,

$$K'_n := 1 + \begin{pmatrix} \wp^{n_2} & \wp^{n_1} \\ \wp^{n_1+1} & \wp^{n_2} \end{pmatrix} \subset K,$$

where $n_1 := [n/2]$ and $n_2 := [(n+1)/2]$ and where $\wp^0 := \mathcal{O}$. Note that $K'_n \subset K'$ is a normal subgroup for all n. We also let Z (resp. Z') denote the cyclic subgroup of G generated by the element $\varpi \in F^\times \subset G$ (resp. by the matrix Π'), where

$$\Pi' := \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}.$$

Note that Z' normalizes K' but not K.

Let τ be a smooth and irreducible (and hence finite-dimensional) representation of K. The *K-level* of τ is the minimal integer $n \geq 1$ such that the restriction of τ to K_n is trivial. The *K-defect* of τ is the integer $n-r$, where r is the minimal integer such that the restriction of τ to U_r has a non-zero fixed vector. Using the filtration (K'_n), we define in a similar way the K'-level and the K'-defect for a smooth irreducible representation of K'.

A smooth and irreducible representation τ of K is called *minimal* if its K-level cannot be lowered by twisting τ with a one-dimensional character. The *minimal level* of τ is the K-level of the twist of τ which is minimal.

Let π be an irreducible admissible supercuspidal representation of G. For short we say that π is a *supercuspidal*. Then there exists a subgroup $J \subset G$, which contains and is compact modulo the center of G, and a finite-dimensional representation σ of J such that π is the compactly induced representation c-$\mathrm{Ind}_J^G(\sigma)$. Furthermore, the pair (J, σ) can be chosen in a very specific way, as follows. We distinguish two cases. In the first case, J is the group ZK. Then the supercuspidal π is called *unramified*. The restriction of σ to K is called the *K-type* of π and is denoted by τ. In the second case $J = Z'K'$ and π is called *ramified*. Then the K-type of π is defined as the induced representation

$$\tau := \mathrm{Ind}_{K'}^K(\sigma|_{K'}).$$

In both cases, the pair (J, σ) is called a *type* for π.

We have the following fundamental result of Kutzko [15].

Proposition 4.2 *Suppose that the K-type τ of π is minimal of level n. Then the following holds.*

(i) *If π is unramified then τ has K-defect zero.*

(ii) *If π is ramified, then the restriction of σ to K' has K'-level $2n-2$ and K'-defect zero. Furthermore, $n \geq 2$.*

4.3 The Swan and the discriminant conductor of a type Let π be a supercuspidal representation with K-type τ. Let m be the K-level of τ. Considering τ as a representation of the finite group $G^{(m)} = K/K_m$, we can define the admissible sheaf on $\mathsf{X}(1)$

$$\mathcal{F} := (f_{m*}\bar{\mathbb{Q}}_\ell)[\tau],$$

where $f_m : \mathsf{X}(\varpi^m) \to \mathsf{X}(1)$ is the étale K/K_m-torsor defined in the previous section. Let $x \in \partial\mathsf{X}(1)$ be the unique end of the disk $\mathsf{X}(1)$. The following theorem, which is our first main result, computes the Swan conductor and the discriminant conductor of $\mathcal{F}$ at x in the unramified case.

Theorem 4.3 *Suppose that π is unramified. Let n be the minimal K-level of τ. Then we have*

$$\mathrm{sw}_x(\mathcal{F}) = -\frac{q+1}{q-1} \cdot \dim\tau = -(q+1)q^{n-1}$$

and

$$\delta_x(\mathcal{F}) = \frac{nq-n+1}{q-1} \cdot \dim\tau = (nq-n+1)q^{n-1}.$$

Proof: By Proposition 3.1, the stabilizers of the boundary components of $\mathsf{X}(\varpi^m)$ are contained in the subgroup of G_m of elements of determinant one. Therefore, if τ' is a twist of τ by a one-dimensional character (which factors through the determinant) and $\mathcal{F}'$ is the sheaf corresponding to τ' then $\mathrm{sw}_x(\mathcal{F}) =$

$\mathrm{sw}_x(\mathcal{F}')$ and $\delta_x(\mathcal{F}) = \delta_x(\mathcal{F}')$. We may therefore assume that τ is minimal of level n. For the rest of the proof, we consider τ as a representation of $G^{(n)}$. Let V be the vector space underlying τ. It follows from the construction of τ in [15] that $\dim V = (q-1)q^{n-1}$.

Let $x_n \in \partial\mathsf{X}(\varpi^n)$ be an end whose stabilizer $G^{(n)}_{x_n} \subset G^{(n)}$ consists of upper triangular matrices. Let $\gamma_1 > \ldots > \gamma_{2n-1}$ be the jumps for the filtration of higher ramification groups $G^\gamma \subset G^{(n)}_{x_n}$. By Proposition 3.1 the subgroup $U^{(n)}_{n-1} \subset G^{(n)}$ is contained in G^γ for all $\gamma \geq \gamma_{2n-1}$. Since τ has K-defect zero (Proposition 4.2), the action of $U^{(n)}_{n-1}$ on V has no fixed vector. It follows that the break decomposition of V has a unique break at γ_{2n-1}, i.e. $V = V(\gamma_{2n-1})$. Using (4), (5) and Corollary 3.2 we get

$$\mathrm{sw}_x(\mathcal{F}) = \#\gamma_{2n-1} \cdot \dim V = -(q+1)q^{n-1}$$

and

$$\delta_x(\mathcal{F}) = -\log_q \gamma^\flat_{2n-1} \cdot \dim V = (nq-n+1)q^{n-1}.$$

□

Corollary 4.4 *With notation as in Theorem 4.3, we have*

$$\dim H^1_{\mathrm{p}}(\mathsf{X}(1), \mathcal{F}) = 2q^{n-1}.$$

Proof: The Ogg-Shafarevich formula in [13] gives

$$\sum_{i=0}^{2} (-1)^i \dim H^i_c(\mathsf{X}(1), \mathcal{F}) = \mathrm{rank}\,\mathcal{F} + \mathrm{sw}_x(\mathcal{F}) = -2q^{n-1}.$$

Clearly $H^0_c(\mathsf{X}(1), \mathcal{F}) = 0$. Moreover, $\dim H^2_c(\mathsf{X}(1), \mathcal{F}) = \dim H^0(\mathsf{X}(1), \mathcal{F})$ is equal to the dimension of the space of fixed vectors of the representation τ. Since τ is irreducible and nontrivial, this number is also zero. We conclude that $\dim H^1_c(\mathsf{X}(1), \mathcal{F}) = 2q^{n-1}$. It remains to show that the map $H^1_c(\mathsf{X}(1), \mathcal{F}) \to H^1(\mathsf{X}(1), \mathcal{F})$ is an isomorphism. Indeed, the dimension of the kernel and of the kokernel of this map equals the intertwining number of τ with the trivial representation of the stabilizer $G^{(n)}_y$ of an end $y \in \partial\mathsf{X}(\varpi^n)$. But this number is zero because $G^{(n)}_y$ contains $U^{(n)}$ and τ has K-defect zero. □

Let us now assume that the supercuspidal π is ramified. Let (J, σ) be the type of π, with $J = Z'K'$. We write $\mathsf{X}_0(\varpi) := \mathsf{X}(\varpi)/K'$ for the étale cover of $\mathsf{X}(1)$ corresponding to the subgroup $K' \subset K$. Let $\mathcal{F}'$ denote the admissible sheaf on $\mathsf{X}_0(\varpi)$ corresponding to the restriction of σ to K'. Since the K-type τ of π is the induced representation of the restriction of σ to K', the pushforward of $\mathcal{F}'$ to $\mathsf{X}(1)$ can be identified with the sheaf $\mathcal{F}$.

Let $y \in \partial\mathsf{X}(\varpi)$ be an end of $\mathsf{X}(\varpi)$ such that the stabilizer $G^{(1)}_y \subset G^{(1)}$ of y is equal to the group of upper triangular matrices of determinant one (Proposition 3.1). We see that $\mathsf{X}_0(\varpi)$ has exactly two ends, corresponding to the double cosets

$G_y\backslash K/K'$. (It is not hard to show that $\mathsf{X}_0(\varpi)$ is an open annulus, see [20].) Let $y_1 \in \partial\mathsf{X}_0(\varpi)$ be the image of y and let y_2 the other end of $\mathsf{X}_0(\varpi)$. The next theorem computes the Swan and the discriminant conductor of the sheaf $\mathcal{F}'$ at the two ends y_1 and y_2.

Theorem 4.5 *Suppose that the supercuspidal π is ramified and that its K-type has minimal K-level n. Let $\mathcal{F}'$ be the admissible sheaf on $\mathsf{X}_0(\varpi)$ induced from the type σ of π. Let y_1, y_2 be the two ends of $\mathsf{X}_0(\varpi)$. Then*

$$\mathrm{sw}_{y_1}(\mathcal{F}') = \mathrm{sw}_{y_2}(\mathcal{F}') = -\frac{q+1}{q-1}\cdot \dim\sigma = -(q+1)q^{n-2}$$

and

$$\delta_{y_1}(\mathcal{F}') = \delta_{y_2}(\mathcal{F}') = \frac{nq-q-n}{q-1}\dim\sigma = (nq-q-n)q^{n-2}.$$

Proof: It is shown in [20] that the pair $(\mathsf{X}_0(\varpi), \mathcal{F}')$ has an automorphism which switches the two ends y_1 and y_2. It follows that the Swan and the discriminant conductor of $\mathcal{F}'$ at the two ends are equal. Therefore it suffices to compute $\mathrm{sw}_{y_1}(\mathcal{F}')$ and $\delta_{y_1}(\mathcal{F}')$. Note also that the dimension of σ is equal to $(q-1)q^{n-2}$, by its construction in [15]. By Proposition 4.2, the restriction of σ to K' has K'-level $2n-2$ and K'-defect zero. Therefore, the restriction of σ to U_r contains a fixed vector if and only if $r \geq n-1$. Backed up by all these preliminary remarks, the proof proceeds exactly as the proof of Theorem 4.3. $\Box$

Corollary 4.6 *Let $\mathcal{F}$ be the sheaf on $\mathsf{X}(1)$ corresponding to the K-type τ of a ramified supercuspidal representation π. Let n denote the minimal K-level of τ. Then*

$$\dim H^1_{\mathrm{p}}(\mathsf{X}(1), \mathcal{F}) = 2(q+1)q^{n-2}.$$

Proof: We have already remarked that the sheaf $\mathcal{F}$ is the pushforward of the sheaf $\mathcal{F}'$ from Theorem 4.5 via the map $\mathsf{X}_0(\varpi) \to \mathsf{X}(1)$. Therefore, we have

$$H^i(\mathsf{X}(1), \mathcal{F}) = H^i(\mathsf{X}_0(\varpi), \mathcal{F}')$$

for all i, and the same holds for cohomology with compact support and parabolic cohomology. The Ogg–Shafarevich formula from [13], applied to the open annulus $\mathsf{X}_0(\varpi)$, gives

$$\sum_{i=0}^{2}(-1)^i \dim H^i_c(\mathsf{X}_0(\varpi), \mathcal{F}') = \mathrm{sw}_{y_1}(\mathcal{F}') + \mathrm{sw}_{y_2}(\mathcal{F}') = -2(q+1)q^{n-2},$$

by Theorem 4.5. For the proof that $H^i_c(\mathsf{X}_0(\varpi), \mathcal{F}') = 0$ for $i = 0, 2$ and that $H^1_c(\mathsf{X}_0(\varpi), \mathcal{F}') \cong H^1(\mathsf{X}_0(\varpi), \mathcal{F}')$ one proceeds as in the proof of Corollary 4.4. $\Box$

Remark 4.7 Let $\mathcal{H}$ be the representation of $G \times B^\times \times W_F$ defined in §4.1. Corollary 4.4 and Corollary 4.6 imply that for an irreducible supercuspidal π we have

$$\dim \operatorname{Hom}_G(\pi, \mathcal{H}) \;=\; \begin{cases} 4q^{n-1}, & \text{if } \pi \text{ is unramified,} \\ 2(q+1)q^{n-2}, & \text{if } \pi \text{ is ramified,} \end{cases}$$

where n is the minimal level of π. Indeed, using Frobenius reciprocity and the definition of the type of π we see that $\operatorname{Hom}_G(\pi, \mathcal{H})$ can be identified with (a) the direct sum of two copies of $\operatorname{Hom}_K(\tau, \mathcal{H}_0) = H^1_{\mathrm{p}}(\mathsf{X}(1), \mathcal{F})$ if π is unramified and with (b) $\operatorname{Hom}_{K'}(\sigma, \mathcal{H}_0) = H^1_{\mathrm{p}}(\mathsf{X}_0(\varpi), \mathcal{F}')$ if π is ramified. From Theorem 4.1 we conclude that $\dim \mathrm{JL}(\pi) = 2q^{n-1}$ in the unramified and $\dim \mathrm{JL}(\pi) = (q+1)q^{n-2}$ in the ramified case. These formulas can be verified by looking at the explicit construction of supercuspidal representations of $B^\times$ by type theory.

4.4 Only supercuspidals occur in the parabolic cohomology Fix a character $\epsilon : \mathcal{O}^\times \to \bar{\mathbb{Q}}_\ell^\times$ and let n denote the *exponent* of ϵ, i.e. the minimal positive integer n such that ϵ is trivial on $1 + \mathfrak{p}^n$. Let $K_0(n)$ be the subgroup of K consisting of matrices whose lower left entry is divisible by ϖ^n. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\mapsto\; \epsilon(a)$$

defines a character $\tilde{\epsilon} : K_0(n) \to \bar{\mathbb{Q}}_\ell^\times$. Let

$$u = u_n(\epsilon) := \operatorname{Ind}_{K_0(n)}^K \tilde{\epsilon}$$

denote the induced representation. One can show that u is irreducible and has K-level n (see e.g. [6]). Let $\mathcal{F}$ denote the admissible sheaf on $\mathsf{X}(1)$ corresponding to the representation u.

Proposition 4.8 *We have* $H^1_{\mathrm{p}}(\mathsf{X}(1), \mathcal{F}) = 0$.

Proof: Let $x \in \partial\mathsf{X}(1)$ be the unique end. Choose $y \in \partial\mathsf{X}(\varpi^n)$ as in Proposition 3.1. In the following, we shall freely use the notation from the statement of Proposition 3.1 and Corollary 3.2. Let V be the vector space underlying the representation u. Note that $\dim V = [K : K_0(n)] = (q+1)q^{n-1}$. Consider the break decomposition

$$V = \bigoplus_{i=1}^{2n-1} V(\gamma_i)$$

of V induced from the filtration of higher ramification groups at y. It follows from Lemma 4.9 below that $V(\gamma_i) = 0$ for $i = 1, \ldots, n-2$ and

$$\dim V(\gamma_i) = \begin{cases} 2, & i = n-1, \\ q^{j+1} - q^j, & i = n+j,\; j = 0, \ldots, n-2, \\ q^n - 1, & i = 2n-1. \end{cases}$$

Then by (4) and Corollary 3.2 the Swan conductor of $\mathcal{F}$ at x is

$$\begin{aligned}
\mathrm{sw}_y(\mathcal{F}) &= \sum_{i=1}^{2n-1} \#\gamma_i \cdot \dim V(\gamma_i) \\
&= 2\,(q+1)(1+q+\ldots+q^{n-2}) \;-\; \sum_{j=0}^{n-2} q^j(q+1) \;-\; (q^n-1)\frac{q+1}{q-1} \\
&= (q+1)\big((1+\ldots+q^{n-2}) - (1+\ldots+q^{n-1})\big) \\
&= -(q+1)\,q^{n-1}.
\end{aligned}$$

Therefore, the Ogg-Shafarevich formula gives

$$\dim H^1_c(\mathsf{X}(1), \mathcal{F}) = -\mathrm{sw}_x(\mathcal{F}) - \dim V = 0.$$

$\square$

Lemma 4.9 *For $i = 1, \ldots, n-1$ we have $V^{G^{\gamma_i}} = 0$. For $j = 0, \ldots, n-1$ and $i = n+j$ we have*

$$\dim V^{G^{\gamma_i}} = 1 + q^j.$$

Proof: Left to the reader. $\square$

Corollary 4.10 *Let $\mathcal{H}$ be the representation defined in §4.1. Let π be an irreducible smooth admissible representation of G. If π is not supercuspidal then*

$$\mathrm{Hom}_G(\pi, \mathcal{H}) = 0.$$

Proof: Clearly, if π occurs in the G-representation $\mathcal{H}$ then the restriction of π to the subgroup K occurs in the K-representation $\mathcal{H}_0 = \varinjlim H^1_{\mathrm{p}}(\mathsf{X}(\varpi^n), \bar{\mathbb{Q}}_\ell)$. If π is not supercuspidal then by [3, Appendix], $\pi|_K$ either contains the trivial representation or a representation $u = u_n(\epsilon)$ for some character ϵ of exponent n. But the trivial representation does not occur in $\mathcal{H}_0$ because $H^1_{\mathrm{p}}(\mathsf{X}(1), \bar{\mathbb{Q}}_\ell) = 0$ and u does not occur in $\mathcal{H}_0$ by Proposition 4.8. This proves the claim. $\square$

Remark 4.11 Laurent Fargues has explained to me a much better proof of the statement of Corollary 4.10, which also works for Lubin-Tate spaces of arbitrary dimension.

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